**Relativistic Fermions**

**Special Relativity for single particle (review)**

So we’ll recall that our attempt at a relativistically invariant theory for fermions gave us the Dirac equation, which is as follows, using natural units ℏ = 1, c = 1:



(note we’re using the metric η = diag(1, -1, -1, -1). But of course there were issues with this theory. One was that it suffers from negative energy solutions. Another is that it is incapable of describing particle creation/annihilation. And so we look for a *field* theory of these particles. In this view, the particles will emerge as excitations of the field, in the same way that phonons emerge as excitations of an elastic field, and photons as excitations of the EM field.

**Fermionic Field**

We find ourselves in the same position as last time, having an equation of motion for our field, but not an H per se´. But this time its easier to go backwards. We may, from our notion of 2nd quantization, presume that our Hamiltonian is simply the second quantized version of the single particle H:

 

The commutation relation can be easily inferred from the Lagrangian formulation. The Lagrangian from which H follows is:

 

where = ψ†β. We can verify (implicit summation over indices) it’s true. First note the momentum conjugate is (seems odd that we consider ψm to comprise one d.o.f. and not two, since the field can take on complex numbers? well it takes on Grassman numbers hmmm…in any event the FFE does evince just 4 d.o.f. and I guess we can treat these as just constituting the ψm components):



(and so the commutation relations do follow) and then (implicit summation over spin index, m)



And the anti-commutation relations follow from {ψm(**x**), πm´(**x**´)} = iδmm´δ(**x**-**x**´). So there we go. We’ll get to why we anti-commute later I guess. Taking the functional derivative of the action, we will get our Dirac equation of motion again,



Now we want to solve the equation, and construct the free field expansion. We’ll work in the Weyl representation.



To solve this we can write our field as the inverse spatial Fourier transform of, well, the spatial Fourier transform of our field.



Plugging this in, we find:



and so,



We can insert a resolution of identity in terms of the eigenvectors to get a formal solution:



Now we need to work out the eigenvalues/eigenvectors of the RHS. So apropos eigenvalues,



which are what we expect. And the eigenvectors are some kind of (a b), where a and b are both 2D column vectors. The bottom row of our matrix dictates:



Now let k0 = E+ = ωk = √(|**k**|2 + m2), so that we have the four vector kμ = (k0, **k**). And note that:



So then for the positive/negative energy solutions we can say:

 

Looking at the form of our solutions, we can make them more symmetric if we define b+ as √(k·)ξ(s), where ξ(s) is (1 0) if s = +1/2 but is (0 1) if s = -1/2, and b- as -√(k·σ)η(s), where η(s) is also (1 0) or (0 1) under the same spin conditions. Then we have for our eigenvectors:



(the -sign in **k** is for later convenience). Note the the top (particle) and bottom (antiparticle) halves of u and v both come in with the same spin: both up or both down. The eigenvectors are functions, technically, of just **k**, since k0 is determined from it. Note that our eigenvectors are orthogonal of course. Their normalization is (no summation over s):



Now we’ll continue with our solution of the equation. Since we have the eigenvectors/values of our matrix, we can say:



Taking the inverse transform, we have:



Now let’s examine the commutation relations for ψ0(**k**) that are implied by the canonical ones. So, setting t = 0 for convenience, we have, writing out things in terms of indices (implicit summation)



Let’s just verify that:



If so, then we’d have:



which is what we want. Regardless, we can identify the creation/annihilation operators:



(anticipating a sign change we’ll be making later) We want to normalize them, and we can use our verified anticommutation relation above to help,



The b guy should work out the same way. So we have:



So then our free-field expansion works out to:



So,



Let’s plug this into H and see what we get…(setting t = 0 for simplicity). Then we’ll have:



(no *need* to interchange the a’s/u’s and b’s/v’s since the latter are just numbers and they commute with the former)



In the last line we changed variables to -**k** in the last two terms. Continuing, strategically inserting a ββ,



Now use, from Appendix,



So,



Using orthogonality relations between u’s and v’s, we have:



Here, we can see that we need anti-commutation relations to avoid an infinite negative energy particle cascade. And in that case we’d have:



Neglecting the infinite constant, like we had to with the boson field, we simply have:



We interpret the a’s as handling the ‘particle’ and the b’s as handling the anti-particle.

**Conservation Laws**

Let’s go ahead and work out the conserved quantities. We already know that one is H itself. But let’s work out the momentum. We have (implicit summation over m):



where in the last we’re eliding the implicit subscript stuff. We’re off by the usual negative sign, but whatever. From our knowledge of 2nd quantization, we might have anticipated this form for **P**. Anyway, working it out:



Ignoring the stuff that will have zero expectation, we come to:



we can anti-commute the b’s at the cost of an infinite constant (ignored),



and in the last term we can change **k** → -**k**.



Of course the spinor eigenvectors are orthogonal (normalized to 2ωk). So we have, dropping the (-) sign – have to figure out what’s up with that sometime:



as we might expect. Let’s consider angular momentum. Borrowing some work done in the Symmetry file, we have:



where λ is the first order rotation matrix. This is, for spinors:



where recall:



Furthermore, in the same representation we have:



and so,



Filling this in we get:



which is indeed the total angular momentum **J** = **L** + **S** that we know should be conserved. We can, alternatively, explicitly check that **J** commutes with H, but nah.

Like with the complex bosons, we can also observe that the fermion field has U(1) symmetry: the mapping ψm → ψmeiλ ≈ ψm + iλψm leaves L invariant. Carrying over the bosonic calculation, we can work out the corresponding conserved charge density:



and the associated current:



I guess we can ignore the minus signs, and the λ, and say:



and these are just as we calculated in the relativistic QM file. For Q = ∫d3xρ, we’ll get:



**Appendix on some stuff**

Just want to look at some products of spinor and such. First recall from the relativistic quantum mechanics file, that:



In particular,



and our spinor definitions above,



Can see from above that:



And,



And recalling how by definition = a†γ0, we also have that:



Here’s some more stuff,



which is what we expect. Now consider:



and,



We’ll take note that it follows, due to some of the orthogonality relations above, that:

